A characterization of dissimilarity families of trees

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Abstract

Let $\mathcal{T} = (T, w)$ be a weighted finite tree with leaves $1, ..., n$. For any $I := \{i_1, ..., i_k\} \subset \{1, ..., n\}$, let $D_I(\mathcal{T})$ be the weight of the minimal subtree of T connecting $i_1, ..., i_k$; the $D_I(\mathcal{T})$ are called k-weights of $\mathcal{T}$. Let $\{D_I\}_{I \text{ } k\text{-subset of } \{1,...,n\}}$ be a family of real numbers. We say that a weighted tree $\mathcal{T} = (T, w)$ with leaves $1, ..., n$ realizes the family if $D_I(\mathcal{T}) = D_I$ for any k-subset I of $\{1, ..., n\}$.

In this paper we find some equalities and inequalities characterizing the families of real numbers parametrized by the k-subsets of $\{1, ..., n\}$ that are the families of k-weights of weighted trees whose leaf set is equal to $\{1, ..., n\}$ and whose weights of the internal edges are positive (where we say that an edge e is internal if there exists a path with endpoints of degree greater than 2 and containing e).

1 Introduction

For any graph G, let $E(G)$, $V(G)$ and $L(G)$ be respectively the set of the edges, the set of the vertices and the set of the leaves of G. A **weighted graph** $\mathcal{G} = (G, w)$ is a graph G endowed with a function $w : E(G) \to \mathbb{R}$. For any edge e, the real number $w(e)$ is called the weight of the edge. If all the weights are nonnegative (respectively positive), we say that the graph is **nonnegative-weighted** (respectively **positive-weighted**). We say that an edge e is internal if there exists a path with endpoints of degree greater than 2 and containing e. If the weights of the internal edges are nonzero, we say that the graph is **internal-nonzero-weighted** and, if the weights of the internal edges are positive, we say that the graph is **internal-positive-weighted**. For any finite subgraph G' of G, we define $w(G')$ to be the sum of the weights of the edges of G'. In this paper we will deal only with weighted finite trees.

Definition 1. *Let $\mathcal{T} = (T, w)$ be a weighted tree. For any distinct $i_1, ..., i_k \in V(T)$, we define $D_{\{i_1,....,i_k\}}(\mathcal{T})$ to be the weight of the minimal subtree containing $i_1, ..., i_k$. We call this subtree "the subtree realizing $D_{\{i_1,....,i_k\}}(\mathcal{T})$". More simply, we denote $D_{\{i_1,....,i_k\}}(\mathcal{T})$ by $D_{i_1,....,i_k}(\mathcal{T})$ for any order of $i_1, ..., i_k$. We call the $D_{i_1,....,i_k}(\mathcal{T})$ the **k-weights** of $\mathcal{T}$ and we call a k-weight of $\mathcal{T}$ for some k a **multiweight** of $\mathcal{T}$.*

Throughout the paper we will use the following standard notation: for any set Z and $k \in \mathbb{N}$, we denote by $\binom{Z}{k}$ the set of the k-subsets of Z. If Z is a subset of $V(T)$, the k-weights $D_{i_1,...,i_k}(\mathcal{T})$ with $i_1, \dots, i_k \in Z$ give a vector in $\mathbb{R}^{\sharp\binom{Z}{k}}$. This vector is called **k-dissimilarity vector** of $(\mathcal{T}, Z)$. Equivalently, we can

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speak of the **family of the k-weights** of $(\mathcal{T}, Z)$ or of the **k-dissimilarity family of $(\mathcal{T}, Z)$.** The vertices in Z are said **labelled**.
If Z is a finite set, $k \in \mathbb{N}$ and $k < \#Z$, we say that a family of real numbers $\{D_I\}_{I \in \binom{Z}{k}}$ is **treelike** (respectively p-treelike, nn-treelike, inz-treelike, ip-treelike) if there exist a weighted (respectively positive-weighted, nonnegative-weighted, internal-nonzero-weighted, internal-positive-weighted) tree $\mathcal{T} = (T, w)$ and a subset Z of the set of its vertices such that $D_I(\mathcal{T}) = D_I$ for any k-subset I of Z. In this case, we say also that $\mathcal{T}$ realizes the family $\{D_I\}_{I \in \binom{Z}{k}}$. If in addition $Z \subset L(T)$, we say that the family is **l-treelike** (respectively p-l-treelike, nn-l-treelike, inz-l-treelike, ip-l-treelike). In this paper we will consider only the problem of l-treelikeness, that is, we will consider labels only on the leaves.
Weighted graphs have applications in several disciplines, such as biology, psychology, archeology, engineering. Phylogenetic trees are weighted trees whose vertices represent species and the weight of an edge is given by how much the DNA sequences of the species represented by the vertices of the edge differ. There is a wide literature concerning graphlike dissimilarity families and treelike dissimilarity families, in particular concerning methods to reconstruct weighted trees from their dissimilarity families; these methods, for instance the so-called neighbor-joining method, are used by biologists to reconstruct phylogenetic trees. See for example [16], [20] and [8], [17] for overviews. Weighted graphs can also represent hydraulic webs or railway webs where the weight of a line is the difference between the earnings and the cost of the line or the length of the line. It can be interesting, given a family of real numbers, $\{D_{i_1,...,i_k}\}_{i_1,...,i_k}$, to wonder if there exists a weighted tree with it as family of k-weights; moreover the study of the subset of the treelike vectors, and in particular of the equalities and inequalities characterizing it, can be useful if we search a tree whose k-weights have some characteristics, for instance satisfy some given equalities or inequalities. We recall also that the importance of the study of k-weights for $k \geq 3$ is due to the fact that they seem statistically more reliable than 2-weights (see [13] and [19]).
We recall the most important results concerning treelike dissimilarity families.
A criterion for a metric on a finite set to be nn-l-treelike was established in [6], [18], [21]:

Definition 2. *Let $n \in \mathbb{N}$ with $n \geq 4$. Let $\{D_I\}_{I \in \binom{\{1,...,n\}}{2}}$ be a family of positive real numbers. We say that the D_I satisfy the 4-**point condition** if and only if for all distinct $a, b, c, d \in \{1, ..., n\}$, the maximum of*

$$\{D_{a,b} + D_{c,d}, D_{a,c} + D_{b,d}, D_{a,d} + D_{b,c}\}$$

is attained at least twice.

Theorem 3. *Let $n \in \mathbb{N}$ with $n \geq 4$. Let $\{D_I\}_{I \in \binom{\{1,...,n\}}{2}}$ be a family of positive real numbers satisfying the triangle inequalities. It is p-treelike (or nn-l-treelike) if and only if the 4-point condition holds.*

Also the study of general weighted trees can be interesting and, in 1995, Bandelt and Steel proved a result, analogous to Theorem 3, for general weighted trees:

Theorem 4. ***(Bandelt-Steel, [3], Theorem 1)*** *Let $n \in \mathbb{N}$ with $n \geq 4$. For any family of real numbers $\{D_I\}_{I \in \binom{\{1,...,n\}}{2}}$, there exists a weighted tree $\mathcal{T}$ with leaves $1, ..., n$ such that $D_I(\mathcal{T}) = D_I$ for any $I \in \binom{\{1,...,n\}}{2}$ if and only if the so-called* relaxed *4-point condition holds, i.e. for any $a, b, c, d \in \{1, ..., n\}$, at least two among $D_{a,b} + D_{c,d}$, $D_{a,c} + D_{b,d}$, $D_{a,d} + D_{b,c}$ are equal.*

An easy variant of the theorems above is the following:

Theorem 5. *Let $n \in \mathbb{N}$ with $n \geq 4$. For any family of real numbers $\{D_I\}_{I \in \binom{\{1,\dots,n\}}{2}}$, there exists an internal-positive weighted tree $\mathcal{T}$ with leaves $1, \dots, n$ such that $D_I(\mathcal{T}) = D_I$ for any $I \in \binom{\{1,\dots,n\}}{2}$ if and only if the 4-point condition holds.*

In fact, if the 4-point condition holds, in particular the relaxed 4-point condition holds, so by Theorem 4, there exists a weighted tree $\mathcal{T}$ with leaves $1, \dots, n$ and with 2-weights equal to the D_I; it is easy to see that, since the 4-point condition holds, the weights of the internal edges of $\mathcal{T}$ are nonnegative; by contracting the internal edges of weight 0, we get an ip-weighted tree with leaves $1, \dots, n$ and with 2-weights equal to the D_I.
For higher k the literature is more recent, see [1], [4], [9], [10], [11], [12], [13], [14], [15]. Three of the most important results for k-weights with $k \geq 3$ are the following.

Theorem 6. ***(Herrmann, Huber, Moulton, Spillner, [9], Theorem 2)****. Let $n, k \in \mathbb{N} - \{0\}$. If $n \geq 2k$, a family of positive real numbers $\{D_I\}_{I \in \binom{\{1,\dots,n\}}{k}}$ is ip-l-treelike if and only if its restriction to every $2k$-subset of $\{1, \dots, n\}$ is ip-l-treelike.*

Theorem 7. ***(Pachter-Speyer, [13], Theorem 1)****. Let $k, n \in \mathbb{N}$ with $3 \leq k \leq (n+1)/2$. A positive-weighted tree $\mathcal{T}$ with leaves $1, \dots, n$ and no vertices of degree 2 is determined by the values $D_I(\mathcal{T})$, where I varies in $\binom{\{1,\dots,n\}}{k}$.*

Before stating the last theorem, we need some notation. We recall that a quartet on the set $\{1, \dots., n\}$ is a bipartition of a 4-subset of $\{1, \dots., n\}$; we denote by $a, b \,|\, c, d$ the partition of $\{a, b, c, d\}$ into the subsets $\{a, b\}$ and $\{c, d\}$; for any tree T with leaf set $\{1, \dots., n\}$, we say that a quartet $a, b \,|\, c, d$ on $\{1, \dots., n\}$ is displayed by T, or that $\langle a, b \,|\, c, d \rangle$ holds, if the path between a and b and the path between c and d are disjoint; the quartet system of T is defined to be the set of the quartets on $\{1, \dots, n\}$ displayed by T. Moreover, we recall the following definitions from [11]:

Definition 8. • *For any tree T and any natural number s, let us define $T_{\leq s}$ to be the subforest of T whose edge set consists of edges dividing $L(T)$ into two subsets such that at least one of them has cardinality less than or equal to s.*
• *Let $n, k \in \mathbb{N}$ with $k \leq n$. Let $\{D_I\}_{I \in \binom{\{1,\dots.,n\}}{k}}$ be a family of real numbers. For any distinct $i, j \in \{1, \dots., n\}$, define*

$$S_{i,j} = \sum_{Y \in \binom{\{1,\dots.,n\} - \{i,j\}}{k-2}} D_{i,j,Y}.$$

Theorem 9. ***(Levy-Yoshida-Pachter, [11], Theorem 6)*** *Let $n, k \in \mathbb{N}$ with $k \leq n$. Let $\mathcal{T} = (T, w)$ be a positive-weighted tree with $L(T) = \{1, \dots, n\}$ and without vertices of degree 2. Let $S_{i,j}$ for distinct $i, j \in \{1, \dots., n\}$ be defined from the $D_I(\mathcal{T})$ for $I \in \binom{\{1,\dots.,n\}}{k}$ as in Definition 8.*
Then the $S_{i,j}$ satisfy the 4-point condition, so (by Theorem 5) there exists a (unique) ip-weighted tree $\mathcal{T}' = (T', w')$ without vertices of degree 2 such that $D_{i,j}(\mathcal{T}') = S_{i,j}$ for all distinct $i, j \in \{1, \dots, n\}$.
Moreover, the quartet system of T' is contained in the quartet system of T, so T' is obtained from T by contracting some edges (see Theorem 1 in [5]).
Finally, we have that $T_{\leq n-k}$ and $T'_{\leq n-k}$ are isomorphic.

It is easy to see that the theorem holds also if $\mathcal{T}$ is ip-weighted. Moreover Levy, Yoshida and Pachter proposed a neighbor-joining algorithm for reconstructing trees from k-weights. It is natural to wonder if the 4-point condition for the $S_{i,j}$ and some other possible conditions could be sufficient for a family $\{D_I\}_{I \in \binom{\{1,\dots,n\}}{k}}$ to be l-treelike. An easy argument about the numbers of the k-weights, the numbers of the equations given by the 4-point condition and the numbers of edges of a tree with n leaves suggests that the 4-point condition for the $S_{i,j}$ cannot be sufficient to characterize l-treelike families and in the literature we don't find a set of equalities and inequalities characterizing l-treelike families. In this paper, by using the $S_{i,j}$ defined by Levy, Yoshida and Pachter, we find some equalities and inequalities characterizing the families of real numbers parametrized by $\binom{\{1,\dots,n\}}{k}$ that are the families of k-weights of ip-weighted trees with leaf set equal to $\{1, ..., n\}$, see Theorem 20.

2 Notation and some recalls

Notation 10. • *For any $n \in \mathbb{N}$ with $n \geq 1$, let $[n] = \{1, ..., n\}$.*
• *Let S be a set and $f : S \to \mathbb{R}$ be a function. For any A, B subsets of S and any $a, b \in \mathbb{R}$, we denote $a \sum_{x \in A} f(x) + b \sum_{x \in B} f(x)$ by*

$$\left(a \sum_{x \in A} + b \sum_{x \in B} \right) f(x).$$

• *For any set S and any $i \in S$ and $X \subset S$, we write iX instead of $\{i\} \cup X$.*
• *For any $n, k \in \mathbb{N}$ and for any family $\{D_I\}_{I \in \binom{[n]}{k}}$ of real numbers, we denote $D_{\{i_1,....,i_k\}}$ by $D_{i_1,....,i_k}$ for any order of $i_1, ..., i_k$.*
• *Throughout the paper, the word "tree" will denote a finite tree.*
• *A **node** of a tree is a vertex of degree greater than 2.*
• *Let F be a leaf. Let N be the node such that the path p between N and F does not contain any node apart from N. We say that p is the **twig** associated to F. We say that an edge is **internal** if it is not an edge of a twig. It is easy to see that this definition is equivalent to the one we have given in the introduction, which is perhaps less intuitive.*
• *For any tree T, we denote by $\mathring{E}(T)$ the set of the internal edges of T.*
• *We say that a tree is **essential** if it has no vertices of degree 2.*
• *If a and b are vertices of a tree, we denote by $p(a, b)$ the path between a and b.*
• *Let T be a tree and let S be a subset of $L(T)$. We denote by $T|_S$ the minimal subtree of T whose set of vertices contains S. Let $\tilde{E}(T|_S) = \mathring{E}(T) \cap E(T|_S)$. Observe that in general $\tilde{E}(T|_S) \neq \mathring{E}(T|_S)$, see Figure 1 for an example. If $\mathcal{T} = (T, w)$ is a weighted tree, we denote by $\mathcal{T}|_S$ the tree $T|_S$ with the weighting induced by w.*
• *Let T be a tree and let $a, b, c, d, x \in L(T)$. Let S be a subtree of $T|_{a,b,c,d}$.*
*Let $\tilde{x}$ be the vertex such that $p(x, \tilde{x})$ is the minimal path whose union with $T|_{a,b,c,d}$ is connected; we say that x **clings** to $T|_{a,b,c,d}$ in S if $\tilde{x} \in V(S)$.*
See Figure 2 for an example: let T be the tree in the figure and let $S = p(a, b)$.

Definition 11. *Let T be a tree. We say that two leaves i and j of T are **neighbours** if in $p(i, j)$ there is only one node; furthermore, we say that $C \subset L(T)$ is a **cherry** if any $i, j \in C$ are neighbours.*
*The **stalk** of a cherry is the unique node in the path with endpoints any two elements of the cherry.*

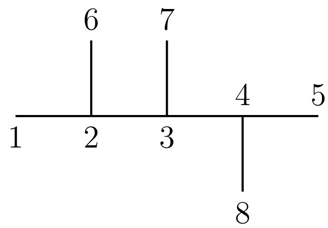

Figure 1: Let T be the tree in the figure and let $S = \{1, 5, 7, 8\}$. The edge $\{2, 3\}$ is in $\tilde{E}(T|_S)$ but not in $\mathring{E}(T|_S)$.

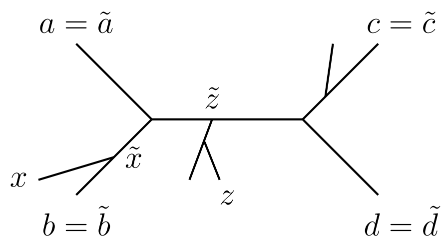

Figure 2: The leaves x, a, b cling to $T|_{a,b,c,d}$ in $S := p(a, b)$, while z, c, d do not cling to $T|_{a,b,c,d}$ in S

Let $a, b, c, d \in L(T)$. In the Introduction we have said that $\langle a, b \,|\, c, d\rangle$ holds if the path between a and b and the path between c and d are disjoint; this is equivalent to say that in $T|_{\{a,b,c,d\}}$ we have that a and b are neighbours, c and d are neighbours, and a and c are not neighbours; in this case we denote by $\gamma_{a,b,c,d}$ the path between the stalk $s_{a,b}$ of $\{a, b\}$ and the stalk $s_{c,d}$ of $\{c, d\}$ in $T|_{\{a,b,c,d\}}$; we call it the ***bridge*** *of the 4-set $\{a, b, c, d\}$. The symbol $\langle a, b \,|\, c, d\rangle$ is called* ***Buneman's index*** *of $\{a, b, c, d\}$.*

Definition 12. *Let $n, k \in \mathbb{N} - \{0\}$. We say that a tree P is a* ***pseudostar*** *of kind (n, k) if $\#L(P) = n$ and any edge of P divides $L(P)$ into two sets such that at least one of them has cardinality greater than or equal to k. See Figure 13 for an example.*

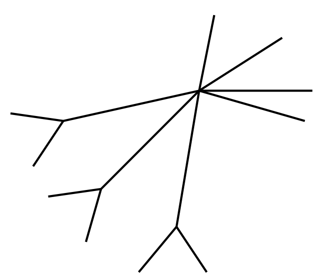

Figure 3: A pseudostar of kind $(10, 8)$

Remark 13. *(i) Let $n \in \mathbb{N} - \{0\}$. A pseudostar of kind $(n, n-1)$ is a star, that is, a tree with only one node.*

(ii) Let $k, n \in \mathbb{N} - \{0\}$. If $\frac{n}{2} \geq k$, then every tree with n leaves is a pseudostar of kind (n,k), in fact, if we divide a set with n elements into two parts, at least one of them has cardinality greater than or equal to $\frac{n}{2}$, which is greater than or equal to k.
(iii) Let $k, n \in \mathbb{N} - \{0\}$. A tree T with n leaves is a pseudostar of kind (n,k) if and only if T is isomorphic to $T_{\leq n-k}$.

Theorem 14. ***(Baldisserri-Rubei, [2], Theorem 16)*** *Let $n, k \in \mathbb{N}$ with $3 \leq k \leq n-1$. Let $\{D_I\}_{I \in \binom{[n]}{k}}$ be a family of real numbers. If it is l-treelike, then there exists exactly one internal-nonzero-weighted essential pseudostar $\mathcal{P}$ of kind (n,k) realizing the family.*
If the family $\{D_I\}_{I \in \binom{[n]}{k}}$ is p-l-treelike, then $\mathcal{P}$ is positive-weighted.

3 Characterization of treelike families

Before stating and proving our theorem, we need to recall some notation and facts from [11] and to state some new lemmas.

Definition 15. *Let $n \in \mathbb{N}$ with $n \geq 4$ and let $\{S_{i,j}\}_{\{i,j\} \in \binom{[n]}{2}}$ be a family of real numbers. For any distinct $a, b, c, d \in [n]$, define*

$$L_{\{a,b\}}^{\{a,b,c,d\}} = \left\{ x \in [n] - \{a,b,c,d\} \;\middle|\; \begin{array}{ll} \text{either} & S_{x,z} - S_{a,z} \text{ does not depend on } z \in \{b,c,d\} \\ \text{or} & S_{x,z} - S_{b,z} \text{ does not depend on } z \in \{a,c,d\} \end{array} \right\} \cup \{a,b\}.$$

We will denote $L_{\{a,b\}}^{\{a,b,c,d\}}$ simply by $L_{a,b}^{a,b,c,d}$ and we will omit the superscript when the 4-set which we are referring to is clear from the context.

The definition above seems rather obscure, but the following proposition and example will clarify why we have introduced it.

Proposition 16. *Let $n \in \mathbb{N}$ with $n \geq 4$ and let $\mathcal{A} = (A, w)$ be an internal-positive-weighted essential tree with $L(A) = [n]$. Denote the 2-weights of $\mathcal{A}$ by $S_{i,j}$ for distinct $i, j \in [n]$. Let $a, b, c, d \in [n]$.*
1) If $\langle a, b \,|\, c, d \rangle$ holds, we have that $L_{a,b}$ is the set of the elements x of $[n]$ clinging to $A|_{a,b,c,d}$ in $p(a,b)$ and $L_{c,d}$ is the set of the elements x of $[n]$ clinging to $A|_{a,b,c,d}$ in $p(c,d)$.
2) We have that $\langle a, b \,|\, c, d \rangle$ holds and the bridge of $\{a,b,c,d\}$ is given by exactly one edge if and only if the following conditions hold:
(i) $S_{a,b} + S_{c,d} < S_{a,c} + S_{b,d} = S_{a,d} + S_{b,c}$,
(ii) $L_{a,b} \cup L_{c,d} = [n]$.

Proof. 1) Observe that $L_{a,b}$ is the set of the elements x of $[n]$ that are neighbours either of a or of b in $A|_{a,b,c,d,x}$; hence, if $\langle a, b \,|\, c, d \rangle$ holds, we have that $L_{a,b}$ is the set of the elements x of $[n]$ clinging to $A|_{a,b,c,d}$ in $p(a,b)$.
2) $\Longrightarrow$ Suppose $\langle a, b \,|\, c, d \rangle$ holds and the bridge of $\{a,b,c,d\}$ is given by exactly one edge; then the weight of the bridge is positive, so (i) holds; moreover, $L_{a,b}$ is the set of the elements x of $[n]$ clinging in $p(a,b)$ and $L_{c,d}$ is the set of the elements x of $[n]$ clinging in $p(c,d)$. So (ii) must hold.
$\Longleftarrow$ If $\langle a, b \,|\, c, d \rangle$ did not hold, then either $A|_{a,b,c,d}$ would be a star or one of $\langle a, c \,|\, b, d \rangle$ and $\langle a, d \,|\, b, c \rangle$ would hold. So we would have either $S_{a,b} + S_{c,d} = S_{a,d} + S_{b,c}$ or $S_{a,b} + S_{c,d} = S_{a,c} + S_{b,d}$, which is absurd

by assumption (i). Hence $\langle a, b \,|\, c, d\rangle$ holds. Moreover, if the bridge of $\{a, b, c, d\}$ were given by more than one edge, then, since A is essential, there would exist $x \in [n]$ clinging in the bridge, and so we would have $x \notin L_{a,b} \cup L_{c,d}$, which is absurd by condition (ii). $\square$

Example. Let $\mathcal{A}$ be the tree represented in Figure 4 with all the weights of the edges equal to 1; consider the 4-set $\{1, 3, 4, 7\}$; we have that $\langle 1, 3 \,|\, 4, 7\rangle$ holds, $L_{1,3}^{1,3,4,7} = \{1, 2, 3, 9, 10\}$ and $L_{4,7}^{1,3,4,7} = \{4, 5, 6, 7, 8\}$, so $L_{1,3}^{1,3,4,7} \cup L_{4,7}^{1,3,4,7} = [10]$ and $\gamma_{1,3,4,7}$ is composed by only one edge. Now consider the 4-set $\{1, 9, 4, 7\}$; we have that $\langle 1, 9 \,|\, 4, 7\rangle$ holds, $L_{1,9}^{1,9,4,7} = \{1, 2, 9, 10\}$ and $L_{4,7}^{1,9,4,7} = \{4, 5, 6, 7, 8\}$, so $L_{1,9}^{1,9,4,7} \cup L_{4,7}^{1,9,4,7} \neq [10]$ and in fact $\gamma_{1,9,4,7}$ is composed by more than one edge.

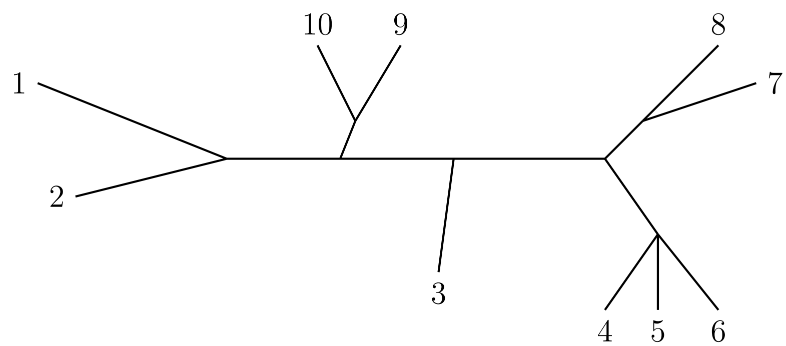

Figure 4: An example to explain Proposition 16

Remark 17. *Let $n, k \in \mathbb{N} - \{0\}$ with $k < n$. Let $\mathcal{T} = (T, w)$ be a weighted essential tree with $L(T) = [n]$. For any $i \in [n]$, we denote the twig associated to i by e_i. Then*

$$w(e_i) = \frac{D_I(\mathcal{T})}{k} - \frac{1}{k} \sum_{e \in \tilde{E}(T|_I)} w(e) + \frac{1}{k} \sum_{j \in I} \left(D_{i,X}(\mathcal{T}) - D_{j,X}(\mathcal{T}) - \sum_{e \in \tilde{E}(T|_{iX})} w(e) + \sum_{e \in \tilde{E}(T|_{jX})} w(e) \right)$$

for any $i \in [n]$, $I \in \binom{[n]}{k}$ and $X = X_{i,j} \in \binom{[n]-\{i,j\}}{k-1}$ (depending on i and j).

Proof. Let $I \in \binom{[n]}{k}$. Then

$$D_I(\mathcal{T}) = \sum_{i \in I} w(e_i) + \sum_{e \in \tilde{E}(T|_I)} w(e). \tag{1}$$

Thus, for any $i, j \in [n]$,

$$w(e_j) - w(e_i) = D_{j,X}(\mathcal{T}) - D_{i,X}(\mathcal{T}) - \sum_{e \in \tilde{E}(T|_{jX})} w(e) + \sum_{e \in \tilde{E}(T|_{iX})} w(e) \tag{2}$$

for any $X \in \binom{[n]-\{i,j\}}{k-1}$. Obviously, for any $i \in [n]$ and any $I \in \binom{[n]}{k}$, we have:

$$k\, w(e_i) = \sum_{j \in I} (w(e_i) - w(e_j)) + \sum_{j \in I} w(e_j). \tag{3}$$

From (1), (2) and (3), we get easily our assertion. $\square$

Proposition 18. ***(Levy-Yoshida-Pachter, Lemma 12.)*** *Let $n, k \in \mathbb{N} - \{0\}$ with $k \leq n$. Let $\mathcal{T} = (T, w)$ be an essential ip-weighted tree with $L(T) = [n]$. As we have already said (see Theorem 9), there exists a (unique) essential ip-weighted tree $\mathcal{T}' = (T', w')$ with $L(T') = [n]$ and with 2-weights the $S_{i,j}$, where the $S_{i,j}$ for distinct $i, j \in [n]$ are defined from the $D_I(\mathcal{T})$ for $I \in \binom{[n]}{k}$ as in Definition 8; the tree T' is obtained from T by contracting some edges. Let e be an internal edge of T' and let $a, b, c, d \in [n]$ such that $\langle a, b \,|\, c, d\rangle$ holds and the bridge of $\{a, b, c, d\}$ is given only by the edge e; then*

$$w(e) = \frac{2w'(e)}{\binom{\#L_{a,b}-2}{k-2} + \binom{\#L_{c,d}-2}{k-2}}. \tag{4}$$

Proposition 19. *Let $n, k \in \mathbb{N} - \{0\}$ with $k \leq n$. Let $\mathcal{T} = (T, w)$ be an essential ip-weighted tree with $L(T) = [n]$. Let $S_{i,j}$ for distinct $i, j \in [n]$ be defined from the $D_I(\mathcal{T})$ for $I \in \binom{[n]}{k}$ as in Definition 8. Let $\mathcal{T}' = (T', w')$ be the essential ip-weighted tree with $L(T') = [n]$ and with 2-weights the $S_{i,j}$. Then T' is a pseudostar of kind (n, k) (so T' is isomorphic to $T'_{\leq n-k}$, which is isomorphic to $T_{\leq n-k}$ by Theorem 9).*

Proof. Suppose, contrary to our claim, that there exists an edge e of T' dividing $L(T') = [n]$ into two parts both of cardinality less than k. By Theorem 9, or more precisely by the analogous statement for ip-weighted trees, the quartet system of T' is contained in the quartet system of T, so T' is obtained from T by contracting some edges (see Theorem 1 in [5]); thus e corresponds to an edge of T dividing $L(T) = [n]$ into two parts both of cardinality less than k. We can suppose e is $\gamma_{a,b,c,d}$ for some $a, b, c, d \in [n]$ such that $\langle a, b \,|\, c, d\rangle$ holds; denote $s_{a,b}$ by t ad $s_{c,d}$ by s (see Definition 11 for the definitions of $\gamma_{a,b,c,d}$ and $s_{a,b}$). We want to show that

$$S_{a,b} + S_{c,d} = S_{a,c} + S_{b,d} \tag{5}$$

(which is absurd since it implies that the weight of e is equal to 0). Obviously $S_{a,b}$ is equal to

$$\sum_{E \in \binom{[n]-\{a,b,c,d\}}{k-2}} D_{a,b,E}(\mathcal{T}) + \sum_{E \in \binom{[n]-\{a,b,c,d\}}{k-3}} D_{a,b,c,E}(\mathcal{T}) + \sum_{E \in \binom{[n]-\{a,b,c,d\}}{k-3}} D_{a,b,d,E}(\mathcal{T}) + \sum_{E \in \binom{[n]-\{a,b,c,d\}}{k-4}} D_{a,b,c,d,E}(\mathcal{T})$$

and analogously $S_{c,d}$, $S_{a,c}$ and $S_{b,d}$. Hence (5) is equivalent to

$$\sum_{E \in \binom{[n]-\{a,b,c,d\}}{k-2}} (D_{a,b,E}(\mathcal{T}) + D_{c,d,E}(\mathcal{T})) = \sum_{E \in \binom{[n]-\{a,b,c,d\}}{k-2}} (D_{a,c,E}(\mathcal{T}) + D_{b,d,E}(\mathcal{T})). \tag{6}$$

We can write $E \in \binom{[n]-\{a,b,c,d\}}{k-2}$ as disjoint union of $E_a, E_b, E_c, E_d, E_t, E_s$, where:
$E_a = \{x \in E |\; x \text{ clings to } T|_{a,b,c,d} \text{ in } p(a,t) - \{t\}\}$
and analogously E_b, E_c, E_d,
$E_t = \{x \in E |\; x \text{ clings to } T|_{a,b,c,d} \text{ in } t\}$
and analogously E_s. By our assumption that e divides $L(T) = [n]$ into two parts both of cardinality less than k, we have that $E_a \cup E_b \cup E_t \neq \emptyset$ and $E_c \cup E_d \cup E_s \neq \emptyset$, in fact: define $A = E_a \cup E_b \cup E_t \cup \{a, b\}$ and $B = E_c \cup E_d \cup E_s \cup \{c, d\}$; we have that $E \cup \{a, b, c, d\}$ is the (disjoint) union of A and B, hence $\#(A \cup B) = \#(E \cup \{a, b, c, d\}) = k + 2$; moreover $\#A \leq k - 1$, $\#B \leq k - 1$, therefore $\#A \geq 3$ and $\#B \geq 3$, which gives the desired conclusion.
So we get:

$$D_{a,b,E}(\mathcal{T}) = w(p(a,t)) + w(p(b,t)) + w(e) + w(p(s, \overline{E_c})) + w(p(s, \overline{E_d}))$$

$$+w\big(T|_{E_a,t}-p(a,b)\big)+w\big(T|_{E_b,t}-p(a,b)\big)+w\big(T|_{E_c,s}-p(c,d)\big)+w\big(T|_{E_d,s}-p(c,d)\big),$$

where $\overline{E_c}$ is the vertex of $T|_{E_c,s}\cap p(s,c)$ which is the most far from s and analogously $\overline{E_d}$. We can write $D_{a,c,E}(\mathcal{T})$, $D_{b,d,E}(\mathcal{T})$, $D_{c,d,E}(\mathcal{T})$ in an analogous way and we get that

$$D_{a,b,E}(\mathcal{T})+D_{c,d,E}(\mathcal{T})=D_{a,c,E}(\mathcal{T})+D_{b,d,E}(\mathcal{T}).$$

So (6) holds. □

Theorem 20. *Let $n,k\in\mathbb{N}$ with $4\le k<n$ and let $\{D_I\}_{I\in\binom{[n]}{k}}$ be a family in $\mathbb{R}$. For any distinct $a,b,c,d\in[n]$, define:*

- $R_{a,b,c,d}=\sum_{X\in\binom{[n]-\{a,b,c,d\}}{k-2}}\big(D_{a,c,X}+D_{b,d,X}-D_{a,b,X}-D_{c,d,X}\big)$*;*
- $L^{\{a,b,c,d\}}_{\{a,b\}}=\left\{x\in[n]-\{a,b,c,d\}\;\middle|\;\text{either }\ R_{x,c,b,a}=R_{x,d,b,a}=0\ \text{ or }\ R_{x,c,a,b}=R_{x,d,a,b}=0\right\}\cup\{a,b\}$*;* *we denote $L^{\{a,b,c,d\}}_{\{a,b\}}$ simply by $L^{a,b,c,d}_{a,b}$;*
- $l_{a,b,c,d}=\binom{\#L^{a,b,c,d}_{a,b}-2}{k-2}+\binom{\#L^{a,b,c,d}_{c,d}-2}{k-2}$*;*
- $Q=\left\{(a,b,c,d)\text{ ordered 4-subset of }[n]\;\middle|\;R_{a,b,c,d}>0,\quad R_{a,d,c,b}=0,\quad L^{a,b,c,d}_{a,b}\cup L^{a,b,c,d}_{c,d}=[n]\right\}/\sim,$ *where $(a,b,c,d)\sim(a',b',c',d')$ if and only if $\{L^{a,b,c,d}_{a,b},L^{a,b,c,d}_{c,d}\}=\{L^{a',b',c',d'}_{a',b'},L^{a',b',c',d'}_{c',d'}\}$;*
- $Q(W)=\{[(a,b,c,d)]\in Q\mid W\cap L^{a,b,c,d}_{a,b}\neq\emptyset,\ W\cap L^{a,b,c,d}_{c,d}\neq\emptyset\}$ *for any $W\in\binom{[n]}{k}$.*

The family $\{D_I\}_{I\in\binom{[n]}{k}}$ is ip-l-treelike if and only if the following conditions hold:
(i) for any $a,b,c,d\in[n]$ with $a<b<c<d$, we have that at least one of $R_{a,b,c,d}$, $R_{a,d,b,c}$, $R_{a,c,d,b}$ is zero; if $R_{x,y,z,w}$ is one of the above and it is zero, then $R_{x,w,y,z}\ge 0$;
(ii) for any distinct $x,y,z,w\in[n]$ such that $R_{x,y,z,w}>0$ and $L^{x,y,z,w}_{x,y}\cup L^{x,y,z,w}_{z,w}=[n]$, then either $\#L^{x,y,z,w}_{x,y}\ge k$ or $\#L^{x,y,z,w}_{z,w}\ge k$;
(iii) for any $I\in\binom{[n]}{k}$, we have:

$$D_I-D_{[k]}+\frac{1}{k}\sum_{i\in I,\,j\in[k]}(D_{j,X}-D_{i,X})=\left[\frac{1}{k}\sum_{i\in I,\,j\in[k]}\left(\sum_{q\in Q(jX)}-\sum_{q\in Q(iX)}\right)-\sum_{q\in Q([k])}+\sum_{q\in Q(I)}\right]\frac{R_q}{l_q}$$

where $X=X_{i,j}$ is the set of the first $k-1$ elements of $[k+1]-\{i,j\}$.

Remark 21. *Observe that*

$$R_{a,b,c,d}=S_{a,c}+S_{b,d}-S_{a,b}-S_{c,d}.$$

So we can see easily that condition (i) is equivalent to the 4-point condition for the $S_{i,j}$.
Finally, observe that the set $L^{a,b,c,d}_{a,b}$ defined in Theorem 20 is exactly the set defined in Definition 15.

Proof of Theorem 20. We will omit the superscript in $L^{a,b,c,d}_{a,b}$ when the 4-set which we are referring to is clear from the context.
$\Longrightarrow$ Let $\mathcal{T}=(T,w)$ be an ip-weighted tree with $L(T)=[n]$ and realizing the family. By Theorem 14, we can suppose that it is an essential pseudostar of kind (n,k).

As we have already said in Remark 21, condition (i) is equivalent to the 4-point condition for the $S_{i,j}$, which holds by Theorem 9.
Let us prove (ii). Let x, y, z, w be distinct elements of $[n]$ such that $R_{x,y,z,w} > 0$, that is $S_{x,z} + S_{y,w} > S_{x,y} + S_{z,w}$, and $L^{x,y,z,w}_{x,y} \cup L^{x,y,z,w}_{z,w} = [n]$; by condition (i), we have that $S_{x,z} + S_{y,w} = S_{x,w} + S_{y,z}$. Let $\mathcal{T}' = (T', w')$ be the ip-weighted essential tree with $L(T') = [n]$ and such that the 2-weights are equal to the $S_{i,j}$; it is a pseudostar of kind (n, k) by Proposition 19. By Proposition 16 part 2), we have that $\langle x, y \,|\, z, w\rangle$ holds and the bridge of $\{x, y, z, w\}$ is given by exactly one edge; moreover, by Proposition 16 part 1), the set $L_{x,y}$ is the set of the leaves clinging to $T'|_{x,y,z,w}$ in $p(x, y)$ and the set $L_{z,w}$ is the set of the leaves clinging to $T'|_{x,y,z,w}$ in $p(z, w)$; so, since $\mathcal{T}' = (T', w')$ is a pseudostar of kind (n, k), we must have either $\#L_{x,y} \geq k$ or $\#L_{z,w} \geq k$, so we get condition (ii).
Let us prove (iii). For any $i \in [n]$, let e_i be the twig of T associated to i. We have:

$$D_I(\mathcal{T}) = \sum_{i\in I} w(e_i) + \sum_{e\in \tilde{E}(T|_I)} w(e) =$$

$$= \sum_{i\in I} \left(\frac{D_{[k]}(\mathcal{T})}{k} - \frac{1}{k} \sum_{e\in\tilde{E}(T|_{[k]})} w(e) + \frac{1}{k}\sum_{j\in[k]} \left(D_{i,X}(\mathcal{T}) - D_{j,X}(\mathcal{T}) - \sum_{e\in\tilde{E}(T|_{iX})} w(e) + \sum_{e\in\tilde{E}(T|_{jX})} w(e) \right)\right) +$$
$$+ \sum_{e\in\tilde{E}(T|_I)} w(e) =$$

$$= D_{[k]}(\mathcal{T}) - \sum_{e\in\tilde{E}(T|_{[k]})} w(e) + \frac{1}{k} \sum_{i\in I,\, j\in[k]} \left(D_{i,X}(\mathcal{T}) - D_{j,X}(\mathcal{T}) - \sum_{e\in\tilde{E}(T|_{iX})} w(e) + \sum_{e\in\tilde{E}(T|_{jX})} w(e) \right) + \sum_{e\in\tilde{E}(T|_I)} w(e),$$

where the second equality holds by Remark 17. Hence

$$\begin{aligned} &\tfrac{1}{k} \textstyle\sum_{i\in I,\, j\in[k]} \left(\sum_{e\in\tilde{E}(T|_{jX})} w(e) - \sum_{e\in\tilde{E}(T|_{iX})} w(e) \right) - \sum_{e\in\tilde{E}(T|_{[k]})} w(e) + \sum_{e\in\tilde{E}(T|_I)} w(e) = \\ &\qquad = D_I(\mathcal{T}) - D_{[k]}(\mathcal{T}) + \tfrac{1}{k} \textstyle\sum_{i\in I,\, j\in[k]} \left(D_{j,X}(\mathcal{T}) - D_{i,X}(\mathcal{T}) \right). \end{aligned} \tag{7}$$

Observe that T and T' are isomorphic: since T and T' are both pseudostars of kind (n, k), we have that T is isomorphic to $T_{\leq n-k}$ and T' is isomorphic to $T'_{\leq n-k}$; moreover $T'_{\leq n-k}$ and $T_{\leq n-k}$ are isomorphic by Theorem 9, so we can conclude that T and T' are isomorphic. By Proposition 16, for any distinct $a, b, c, d \in [n]$, we have that $\langle a, b \,|\, c, d\rangle$ holds in T' and the bridge of $\{a, b, c, d\}$ is given by exactly one edge if and only if $S_{a,b} + S_{c,d} < S_{a,c} + S_{b,d} = S_{a,d} + S_{b,c}$ and $L_{a,b} \cup L_{c,d} = [n]$. Moreover, given $a, b, c, d, a', b', c', d'$ such that $\langle a, b \,|\, c, d\rangle$ holds in T' and the bridge of $\{a, b, c, d\}$ is given by exactly one edge and analogously for a', b', c', d', we have that (a, b, c, d) and (a', b', c', d') give the same edge if and only if they are equivalent for the relation described in the statement of the theorem. So, there is a bijection between Q and $\mathring{E}(T')$, which is equal to $\mathring{E}(T)$. Moreover, for any $W \in \binom{[n]}{k}$, there is a bijection between $Q(W)$ and $\tilde{E}(T'|_W)$, which is equal to $\tilde{E}(T|_W)$; hence, from Proposition 18 and (7), we get condition (iii), in fact, for any $e \in \tilde{E}(T|_W)$, if $q \in Q(W)$ is the element corresponding to e, we have $w(e) = \frac{2w'(e)}{l_q} = \frac{R_q}{l_q}$.
$\Longleftarrow$ Let $\mathcal{T}' = (T', w')$ be an essential ip-weighted tree with 2-weights equal to the $S_{i,j}$ (it exists by condition (i), which is equivalent to 4-point condition for the $S_{i,j}$, and Theorem 5). It is a pseudostar

of kind (n,k) by condition (ii), in fact: let e be an internal edge of T'; let $x,y,z,w \in [n]$ be such that $\langle x,y,\,|\,z,w\rangle$ holds and the bridge of $\{x,y,z,w\}$ is given only by e; then $S_{x,y}+S_{z,w} < S_{x,z}+S_{y,w} = S_{x,w}+S_{y,z}$ and $L_{x,y} \cup L_{z,w} = [n]$; so, by (ii), either $\# L_{x,y} \geq k$ or $\# L_{z,w} \geq k$, and, since $L_{x,y}$ is the set of the leaves clinging to $T'|_{x,y,z,w}$ in $p(x,y)$ and $L_{z,w}$ is the set of the leaves clinging to $T'|_{x,y,z,w}$ in $p(z,w)$, then T' is a pseudostar of kind (n,k).
Let $\mathcal{T}=(T,w)$ be the weighted tree with $T=T'$ and where w is defined as follows: for any $e \in \mathring{E}(T')$, let $a,b,c,d \in [n]$ be such that $\langle a,b\,|\,c,d\rangle$ holds and the bridge of $\{a,b,c,d\}$ is e; define

$$w(e)=\frac{2w'(e)}{\binom{\# L_{a,b}-2}{k-2}+\binom{\# L_{c,d}-2}{k-2}};$$

hence

$$w(e)=\frac{S_{a,c}+S_{b,d}-S_{a,b}-S_{c,d}}{\binom{\# L_{a,b}-2}{k-2}+\binom{\# L_{c,d}-2}{k-2}}=\frac{R_{a,b,c,d}}{l_{a,b,c,d}};$$

moreover, for any $i \in [n]$, define

$$w(e_i)=\frac{D_{[k]}}{k}-\frac{1}{k}\sum_{e\in\tilde{E}(T|_{[k]})} w(e)+\frac{1}{k}\sum_{j\in[k]}\left(D_{i,X}-D_{j,X}-\sum_{e\in\tilde{E}(T|_{iX})} w(e)+\sum_{e\in\tilde{E}(T|_{jX})} w(e)\right)$$

for $X=X_{i,j}\in\binom{[n]-\{i,j\}}{k-1}$ defined as in the statement of the theorem.
We have to show that $D_I(\mathcal{T})=D_I$ for any $I\in\binom{[n]}{k}$. We have:

$$D_I(\mathcal{T})=\sum_{i\in I} w(e_i)+\sum_{e\in\tilde{E}(T|_I)} w(e)=$$

$$=\sum_{i\in I}\left(\frac{D_{[k]}}{k}-\frac{1}{k}\sum_{e\in\tilde{E}(T|_{[k]})} w(e)+\frac{1}{k}\sum_{j\in[k]}\left(D_{i,X}-D_{j,X}-\sum_{e\in\tilde{E}(T|_{iX})} w(e)+\sum_{e\in\tilde{E}(T|_{jX})} w(e)\right)\right)+$$

$$+\sum_{e\in\mathring{E}(T|_I)} w(e)=$$

$$=D_{[k]}-\sum_{e\in\tilde{E}(T|_{[k]})} w(e)+\frac{1}{k}\sum_{i\in I,j\in[k]}\left(D_{i,X}-D_{j,X}-\sum_{e\in\tilde{E}(T|_{iX})} w(e)+\sum_{e\in\tilde{E}(T|_{jX})} w(e)\right)+\sum_{e\in\tilde{E}(T|_I)} w(e),$$

where the second equality holds by the definition of $w(e_i)$. So $D_I(\mathcal{T})=D_I$ if and only if

$$\frac{1}{k}\sum_{i\in I,\,j\in[k]}\left(D_{i,X}-D_{j,X}-\sum_{e\in\tilde{E}(T|_{iX})} w(e)+\sum_{e\in\tilde{E}(T|_{jX})} w(e)\right)+D_{[k]}-D_I-\sum_{e\in\tilde{E}(T|_{[k]})} w(e)+\sum_{e\in\tilde{E}(T|_I)} w(e)=0,$$

that is

$$\frac{1}{k}\sum_{i\in I,\,j\in[k]}\left(\sum_{e\in\tilde{E}(T|_{jX})} w(e)-\sum_{e\in\tilde{E}(T|_{iX})} w(e)\right)-\sum_{e\in\tilde{E}(T|_{[k]})} w(e)+\sum_{e\in\tilde{E}(T|_I)} w(e)=$$

$$= D_I - D_{[k]} + \frac{1}{k} \sum_{i \in I,\, j \in [k]} (D_{j,X} - D_{i,X}),$$

which is true by the definition of the weight of the internal edges and by assumption (iii). □

Remark 22. *Observe that, if $k \leq \frac{n}{2}$, condition (ii) of Theorem 20 is always verified by part (ii) of Remark 13.*

Remark 23. *It is easy to get from Theorem 20 a characterization also for p-l-treelike families. Obviously a family $\{D_I\}_{I \in \binom{[n]}{k}}$ is p-l-treelike if and only if conditions (i), (ii), (iii), (iv) of Theorem 20 hold and, in addition, the number displayed in (iv) is positive for any $i \in [n]$.*

Remark 24. *Observe that condition (ii) of Theorem 20 can be replaced by the following one, which is less elegant, but better from the computational viewpoint:*

$(ii)'$ for any $a, b, c, d \in [n]$ with $a < b < c < d$, if (x, y, z, w) is one of $(a, b, c, d), (a, d, b, c), (a, c, d, b)$ such that $R_{x,y,z,w} > 0$ and $L^{x,y,z,w}_{x,y} \cup L^{x,y,z,w}_{z,w} = [n]$, then either $\# L^{x,y,z,w}_{x,y} \geq k$ or $\# L^{x,y,z,w}_{z,w} \geq k$.

Proof. It is obvious that (ii) implies $(ii)'$. Let us show the converse. In the proof of Theorem 20, we showed that condition (ii) is equivalent to the statement that the ip-weighted essential tree $\mathcal{T}' = (T', w')$ with $L(T') = [n]$ and 2-weights equal to the $S_{i,j}$ is a pseudostar of kind (n, k). So we have to show that also $(ii)'$ implies that T' is a pseudostar of kind (n, k). Let e be an internal edge of T'; let $a, b, c, d \in [n]$ with $a < b < c < d$ be such that e is the bridge of $\{a, b, c, d\}$. Then one of $\langle a, b \,|\, c, d\rangle$, $\langle a, c \,|\, b, d\rangle$, $\langle a, d \,|\, b, c\rangle$ holds. Suppose for instance $\langle a, b \,|\, c, d\rangle$ holds (the other cases are analogous). Then $R_{a,b,c,d} > 0$, $R_{a,d,b,c} < 0$, $R_{a,c,d,b} = 0$. So (x, y, z, w) must be (a, b, c, d) and we have $L^{x,y,z,w}_{x,y} \cup L^{x,y,z,w}_{z,w} = [n]$. By $(ii)'$, either $\# L^{x,y,z,w}_{x,y} \geq k$ or $\# L^{x,y,z,w}_{z,w} \geq k$ and, since $L_{x,y}$ is the set of the leaves clinging to $T'|_{x,y,z,w}$ in $p(x, y)$ and $L_{z,w}$ is the set of the leaves clinging to $T'|_{x,y,z,w}$ in $p(z, w)$, we get that T' is a pseudostar of kind (n, k). □

Example. In the end, we give an example to illustrate Theorem 20. Let $k = 4$ and $n = 6$ and let $\{D_I\}_{I \in \binom{[6]}{4}}$ be the family given by the elements in the first subtable of Table 1; our goal is to establish if there exists an internal-positive-weighted tree $\mathcal{T} = (T, w)$ with $L(T) = [6]$ such that $D_I(\mathcal{T}) = D_I$ for any $I \in \binom{[6]}{4}$. Note that, if the answer is positive, using the proof of Theorem 20, it is possible to construct the pseudostar realizing the family.
We have to check conditions (i), (ii) and (iii) of Theorem 20; in Table 1 and in Table 2 we have calculated the values of the $R_{a,b,c,d}$, the $L^{a,b,c,d}_{a,b}$ and the $Q(\{a, b, c, d\})$ for any a, b, c, d with $1 \leq a < b < c < d \leq 6$. From the second subtable of Table 1, we get easily that condition (i) is satisfied.
Also condition (ii) is satisfied, see for example the first row of Table 1: $R_{1,2,3,4} > 0$, $L^{1,2,3,4}_{1,2} \cup L^{1,2,3,4}_{3,4} = [6]$ and $\sharp L^{1,2,3,4}_{3,4} \geq 4$.
Finally it is easy to check that condition (iii) is satisfied: just as an example let us show that it is satisfied for $I = \{3, 4, 5, 6\}$. The condition is equivalent to the following:

$$D_{1,2,3,4} + D_{3,4,5,6} + \frac{1}{4}(D_{1,2,4,5} + D_{1,2,3,5} - 3D_{2,3,4,5} - 3D_{1,3,4,5} - D_{2,3,4,6} - D_{1,3,4,6} - D_{1,2,4,6} - D_{1,2,3,6}) =$$

$$= \left[\frac{1}{4} \left(\sum_{z \in Q(\{1,2,4,5\})} + \sum_{z \in Q(\{1,2,3,5\})} -3 \sum_{z \in Q(\{2,3,4,5\})} -3 \sum_{z \in Q(\{1,3,4,5\})} - \sum_{z \in Q(\{2,3,4,6\})} - \sum_{z \in Q(\{1,3,4,6\})} + \right. \right.$$

$$- \sum_{z\in Q(\{1,2,4,6\})} - \sum_{z\in Q(\{1,2,3,6\})} \Bigg) + \sum_{z\in Q(\{1,2,3,4\})} + \sum_{z\in Q(\{3,4,5,6\})} \Bigg] \frac{R_z}{l_z};$$

substituting the values of the 4-weights in the first member and calculating the second member we obtain:

$$-15 = \frac{1}{4}\left(-8\frac{R_{1,2,3,4}}{l_{1,2,3,4}} - 8\frac{R_{3,4,5,6}}{l_{3,4,5,6}}\right) + \frac{R_{1,2,3,4}}{l_{1,2,3,4}} + \frac{R_{3,4,5,6}}{l_{3,4,5,6}},$$

that is

$$-15 = -\frac{R_{1,2,3,4}}{l_{1,2,3,4}} - \frac{R_{3,4,5,6}}{l_{3,4,5,6}},$$

which is true. Thus the family $\{D_I\}_I$ is ip-l-treelike; the internal-positive-weighted pseudostar of kind $(6,4)$ realizing the family is the one in Figure 5.

$D_{a,b,c,d}$	$R_{a,b,c,d}$	$R_{a,d,b,c}$	$R_{a,c,d,b}$
$D_{1,2,3,4} = 18$	$R_{1,2,3,4} = 7$	$R_{1,4,2,3} = -7$	$R_{1,3,4,2} = 0$
$D_{1,2,3,5} = 25$	$R_{1,2,3,5} = 7$	$R_{1,5,2,3} = -7$	$R_{1,3,5,2} = 0$
$D_{1,2,3,6} = 27$	$R_{1,2,3,6} = 7$	$R_{1,6,2,3} = -7$	$R_{1,3,6,2} = 0$
$D_{1,2,4,5} = 22$	$R_{1,2,4,5} = 7$	$R_{1,5,2,4} = -7$	$R_{1,4,5,2} = 0$
$D_{1,2,4,6} = 24$	$R_{1,2,4,6} = 7$	$R_{1,6,2,4} = -7$	$R_{1,4,6,2} = 0$
$D_{1,2,5,6} = 23$	$R_{1,2,5,6} = 15$	$R_{1,6,2,5} = -15$	$R_{1,5,6,2} = 0$
$D_{1,3,4,5} = 24$	$R_{1,3,4,5} = 0$	$R_{1,5,3,4} = 0$	$R_{1,4,5,3} = 0$
$D_{1,3,4,6} = 26$	$R_{1,3,4,6} = 0$	$R_{1,6,3,4} = 0$	$R_{1,4,6,3} = 0$
$D_{1,3,5,6} = 25$	$R_{1,3,5,6} = 8$	$R_{1,6,3,5} = -8$	$R_{1,5,6,3} = 0$
$D_{1,4,5,6} = 22$	$R_{1,4,5,6} = 8$	$R_{1,6,4,5} = -8$	$R_{1,5,6,4} = 0$
$D_{2,3,4,5} = 26$	$R_{2,3,4,5} = 0$	$R_{2,5,3,4} = 0$	$R_{2,4,5,3} = 0$
$D_{2,3,4,6} = 28$	$R_{2,3,4,6} = 0$	$R_{2,6,3,4} = 0$	$R_{2,4,6,3} = 0$
$D_{2,3,5,6} = 27$	$R_{2,3,5,6} = 8$	$R_{2,6,3,5} = -8$	$R_{2,5,6,3} = 0$
$D_{2,4,5,6} = 24$	$R_{2,4,5,6} = 8$	$R_{2,6,4,5} = -8$	$R_{2,5,6,4} = 0$
$D_{3,4,5,6} = 19$	$R_{3,4,5,6} = 8$	$R_{3,6,4,5} = -8$	$R_{3,5,6,4} = 0$

Table 1: The $D_{a,b,c,d}$ and the $R_{a,b,c,d}$

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$L_{a,b}^{a,b,c,d}$	$L_{d,c}^{a,b,c,d}$
$L_{1,2}^{1,2,3,4} = \{1,2\}$	$L_{3,4}^{1,2,3,4} = \{3,4,5,6\}$
$L_{1,2}^{1,2,3,5} = \{1,2\}$	$L_{3,5}^{1,2,3,5} = \{3,4,5,6\}$
$L_{1,2}^{1,2,3,6} = \{1,2\}$	$L_{3,6}^{1,2,3,6} = \{3,4,5,6\}$
$L_{1,2}^{1,2,4,5} = \{1,2\}$	$L_{4,5}^{1,2,4,5} = \{3,4,5,6\}$
$L_{1,2}^{1,2,4,6} = \{1,2\}$	$L_{4,6}^{1,2,4,6} = \{3,4,5,6\}$
$L_{1,2}^{1,2,5,6} = \{1,2\}$	$L_{5,6}^{1,2,5,6} = \{5,6\}$
$L_{1,3}^{1,3,4,5} = \{1,2,3\}$	$L_{4,5}^{1,3,4,5} = \{4,5,6\}$
$L_{1,3}^{1,3,4,6} = \{1,2,3\}$	$L_{4,6}^{1,3,4,6} = \{4,5,6\}$
$L_{1,3}^{1,3,5,6} = \{1,2,3,4\}$	$L_{5,6}^{1,3,5,6} = \{5,6\}$
$L_{1,4}^{1,4,5,6} = \{1,2,3,4\}$	$L_{5,6}^{1,4,5,6} = \{5,6\}$
$L_{2,3}^{2,3,4,5} = \{1,2,3\}$	$L_{4,5}^{2,3,4,5} = \{4,5,6\}$
$L_{2,3}^{2,3,4,6} = \{1,2,3\}$	$L_{4,6}^{2,3,4,6} = \{4,5,6\}$
$L_{2,3}^{2,3,5,6} = \{1,2,3,4\}$	$L_{5,6}^{2,3,5,6} = \{5,6\}$
$L_{2,4}^{2,4,5,6} = \{1,2,3,4\}$	$L_{5,6}^{2,4,5,6} = \{5,6\}$
$L_{3,4}^{3,4,5,6} = \{1,2,3,4\}$	$L_{5,6}^{3,4,5,6} = \{5,6\}$

$Q(\{a,b,c,d\})$
$Q(\{1,2,3,4\}) = \{[1,2,3,4]\}$
$Q(\{1,2,3,5\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{1,2,3,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{1,2,4,5\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{1,2,4,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{1,2,5,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{1,3,4,5\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{1,3,4,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{1,3,5,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{1,4,5,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{2,3,4,5\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{2,3,4,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{2,3,5,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{2,4,5,6\}) = \{[1,2,3,4],[3,4,5,6]\}$
$Q(\{3,4,5,6\}) = \{[3,4,5,6]\}$

Table 2: The $L_{a,b}^{a,b,c,d}$ and the $Q(\{a,b,c,d\})$

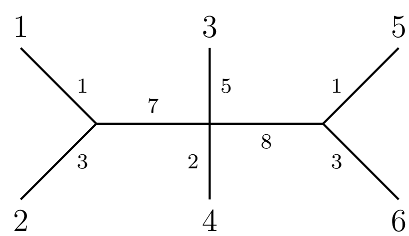

Figure 5: A tree realizing the family

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